Sec 4.3 Linear independence / Bases

Goal: Find "minimal spanning sets"

Q: What constitutes "minimal"?

DEF. Let $\underline{v_1}, \dots, \underline{v_k} \in \mathbb{R}^n$.

1) $\underline{v_1}, \dots, \underline{v_k}$ are linearly dependent if at least one of them is a linear combin. of the others.

2) $\underline{v_1}, \dots, \underline{v_k}$ are lin. indep. if no one of them is a lin. comb. of the others

- When 3 or more vectors involved, "which one" is the linear combination is harder to guess.
- To sidestep this, use the equivalent definition:

DEF (equiv) Try to find scalars $x_1, \dots, x_k$ to make linear combination

$$① \quad x_1 \underline{v_1} + \cdots + x_k \underline{v_k} = \underline{\emptyset} \quad (\underline{\emptyset}_{n\times 1})$$

Always have trivial solution $(x_1, \dots, x_k) = (0, \dots, 0)$

1)* If there is some other $(x_1, \dots, x_k) \neq (0, \dots, 0)$ making $① = \underline{\emptyset}$, then $\underline{v_1}, \dots, \underline{v_k}$ are lin. depen.

2) If no other $(x_1, \dots, x_k)$, $\Longrightarrow$ lin. indep.

* Why? Ex: $2\underline{v_1} - 3\underline{v_2} + 5\underline{v_3} = \underline{\emptyset}$

$$\Rightarrow \underline{v_1} = \tfrac{3}{2}\underline{v_2} - \tfrac{5}{2}\underline{v_3}$$

Ex $\underline{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Are $\underline{u}, \underline{v}, \underline{w}$ lin. indep?

→ Solve for (x_1, x_2, x_3) which make

$$x_1 \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\underline{u}} + x_2 \underbrace{\begin{bmatrix} 7 \\ 8 \end{bmatrix}}_{\underline{v}} + x_3 \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{\underline{w}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \underbrace{\begin{bmatrix} -1 & 7 & 4 \\ 2 & 8 & 3 \end{bmatrix}}_{[\underline{u}\ \underline{v}\ \underline{w}]} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

$$\underline{Ax} = \underline{0}$$

$$\left[\begin{array}{ccc|c} -1 & 7 & 4 & 0 \\ 2 & 8 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 7 & 4 & 0 \\ 0 & 22 & 11 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -7 & -4 & 0 \\ 0 & 1 & 1/2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -1/2 & 0 \\ 0 & \boxed{1} & 1/2 & 0 \end{array}\right]$$

x_3 free → $x_3 = t$

Note that more columns than rows in $[\underline{u}\ \underline{v}\ \underline{w}]$ guarantees free variables

$R_1 \Longrightarrow x_1 = \frac{1}{2}t$

$R_2 \Longrightarrow x_2 = -\frac{1}{2}t$

solution vectors are $\underline{x} = \begin{bmatrix} 1/2\, t \\ -1/2\, t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$\Longrightarrow$ solution set is $\left\{ t \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$

(infinitely many solutions)

Any choice of t gives a combination which $= \underline{0}$

$\underline{Ex}$: • $t = 0 \Rightarrow \underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 0\underline{u} + 0\underline{v} + 0\underline{w} = \underline{0}$ ...

(already knew)

• $t = 1 \Rightarrow \underline{x} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \Rightarrow \frac{1}{2}\underline{u} - \frac{1}{2}\underline{v} + 1\underline{w} = \underline{0}$

$\left(\Rightarrow \underline{w} = -\frac{1}{2}\underline{u} + \frac{1}{2}\underline{v} \right)$

• $t = -3 \Rightarrow \underline{x} = \begin{bmatrix} -3/2 \\ 3/2 \\ -3 \end{bmatrix} \Rightarrow -\frac{3}{2}\underline{u} + \frac{3}{2}\underline{v} - 3\underline{w} = \underline{0}$

So, $\underline{u}, \underline{v}, \underline{w}$ are linearly <u>depen.</u>

Same principle for more than 3 vectors (* see book ex's)

Another way to (sometimes) check depen/indep:

compute determinant

[Read note "independence & determinant"]

<u>Note</u>: Adding/Dropping vectors from a set can change depen/indep, so need to recheck!

"minimal generating set" ⟶ called a "<u>basis</u>"

<u>DEF</u> Set $\{\underline{v}_1, \dots, \underline{v}_n\}$ makes a <u>basis for a set V</u> if <u>both</u> 1) $\operatorname{span}\{\underline{v}_1, \dots, \underline{v}_n\} = V$ 2) $\{\underline{v}_1, \dots, \underline{v}_n\}$ is a lin. indep set.

DEF The <u>standard ordered basis</u> for $\mathbb{R}^n$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\} \overset{DEF}{=} \{ \underline{e_1}, \underline{e_2}, \ldots, \underline{e_n} \}$$

So in $\mathbb{R}^3$, $\{\underline{e_1}, \underline{e_2}, \underline{e_3}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(same as $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$)

in $\mathbb{R}^4$, $\{\underline{e_1}, \underline{e_2}, \underline{e_3}, \underline{e_4}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Sets can have <u>many different bases</u>. How do we find them?

DEF: The <u>dimension</u> of a vec. sp. or subspace of $\mathbb{R}^n$ is the number of <u>linear independent</u> vectors <u>required</u>* to make a basis for the space.

- of course, $\dim(\mathbb{R}^n) = n$

* sets with more or less than $\boxed{\text{dim.}}$ fail either part 1) or 2) of basis defin: (by example)

<u>Ex</u>: Does $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} \right\}$ make a basis of $\mathbb{R}^3$?

<u>No</u>: The set has 4 vectors, but $\dim(\mathbb{R}^3) = 3$.

$\boxed{\begin{array}{l}\star\ (\#\text{ vectors}) > (\text{dim. of space}) \text{ always means} \\ \text{the set is linearly depen.} \\ (\text{"b/c free variables guaranteed"})\end{array}}$

So the above set is not a basis for $\mathbb{R}^3$

Ex : Does $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ $\left(= \{ \underline{e_1}, \underline{e_2}, \underline{e_3} \} \right)$

make a basis for $\mathbb{R}^4$?

4 lin. indep vectors are required to make a basis of $\mathbb{R}^4$, so this is not a basis of $\mathbb{R}^4$...

They are linearly indep tho, so it must be that they do not span $\mathbb{R}^4$.

$\boxed{\begin{array}{l}(\#\text{ of vectors}) < (\text{dimen.}) \Rightarrow \text{set cannot be a} \\ \text{spanning set. May or may not be lin. indep.}\end{array}}$

Ex : Compute a basis for Null $(\underline{A})$, with

$$\underline{A} = \begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix}$$

Last time : $\text{Null}(\underline{A}) = \text{span} \boxed{\left\{ \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$ spanning set for Null $(\underline{A})$

"$\underline{u}$" "$\underline{v}$"

because vectors $\underline{u}, \underline{v}$ are lin. indep. too,

$\{ \underline{u}, \underline{v} \}$ makes a basis for Null $(\underline{A})$.